Research Article

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Some Collision Solutions of the Rectilinear Periodically Forced Kepler Problem

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Abstract: In [5], Ortega has analyzed "generalized" collision solutions of the periodically forced rectilinear Kepler problem. In this note, we explain a different approach to study these solutions by embedding the non-autonomous Hamiltonian system into the zero-energy level of an autonomous Hamiltonian system and by employing the Levi-Civita regularization to regularize the double collisions. In addition to this, under a certain smoothness hypothesis of the periodic force term, we show that there exists a set of positive measure of generalized quasi-periodic solutions in the extended phase space, each of them accumulated by generalized periodic solutions of the system. The energy of these quasi-periodic solutions can have an arbitrary large absolute value.

Keywords: Collision Solutions, Levi-Civita Regularization, Quasi-Periodic Solutions, KAM Theory

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1 Introduction

In the rectilinear periodically forced Kepler problem (studied, e.g., by Lazer and Solimini [3])

$$\ddot{u} = -\frac{1}{u^2} + p(t), \quad p(t + 2\pi) = p(t), \quad u \in \mathbb{R}_+, \tag{1.1}$$

generalized solutions were defined by Ortega in [5] to be collision solutions of the system with collisions regularized by elastic bouncing, i.e., when the particle moves backwardly after the collision while keeping the same energy.

Indeed, in this one-dimensional rectilinear problem, if no collision is admitted, the richness of the dynamics reduces greatly (cf. [5]). In the case when the force term $p(t)$ is C^1, Ortega showed the existence of many generalized periodic solutions by analyzing a Poincaré section associated to the collisions and also showed that the associated Poincaré map is an exact symplectic twist map.

In this note, we explain a different approach to this fact. We embed the non-autonomous Hamiltonian system (1.1) into the zero-energy level of an autonomous Hamiltonian system and we employ the Levi-Civita regularization (see Levi-Civita [4]; note that this already appeared in Goursat [2]) to regularize the collisions. Since the resulting system is regular at collisions, we may then deduce by standard arguments that the Poincaré map associated to the collision set is exact symplectic.

In expressing the unperturbed regularized system in action-angle coordinates, we observe that the twist property of this Poincaré map follows from the non-degeneracy of this system with respect to the action variables in a region where u is supposed to be small enough. When the function $p(t)$ is smooth enough to allow the application of the KAM theorem, we may obtain the following KAM-type results.

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Theorem 1.1. *In* (1.1), *if for some* $\delta > 0$ *the function* $p(t)$ *is* $C^{4+\delta}$, *then there exists a set of positive measure of generalized quasi-periodic solutions in the extended phase space. Moreover, if* $p(t)$ *is* $C^{9+\delta}$, *then there exist infinitely many generalized periodic solutions with frequencies* $\nu_N \in \mathbb{Q}$ *approaching* $\mu \in \mathbb{R} \setminus \mathbb{Q}$, *where* $(\mu, 1)$ *denotes the frequency of a generalized quasi-periodic solution. In addition, these solutions have energies tending to* $-\infty$.

These orbits are obtained from invariant 2-tori lying in the three-dimensional zero-energy hypersurface of the embedded two-degree-of-freedom autonomous system. The existence of these invariant tori prohibits a large change of the corresponding action variables corresponding to the energy and the amplitude of u in (1.1) of the trajectories lying between them. Moreover, by an application of the Poincaré recurrence theorem to the Poincaré map associated to the Poincaré section related to collisions, we conclude that the restricted dynamics lying in the region bounded by two invariant curves (intersections of invariant tori with the Poincaré section) is Poisson stable. When $p(t)$ is sufficiently smooth, this Poisson stability gives, in particular, a partial answer to the question raised in [5] concerning the recurrence of the restricted dynamics.

2 Levi-Civita Regularization of the Rectilinear Kepler Problem

The system (1.1) is a time-periodic Hamiltonian system with Hamiltonian

$$H(u, y, t) = \frac{y^2}{2} - \frac{1}{u} - p(t)u,$$

where y denotes the conjugate momentum of u.

In the case when the perturbation $p(t)$ is of class C^1, we may embed this time-periodic Hamiltonian system in the zero-energy hypersurface $\{\tilde{H} = 0\}$ of the autonomous C^1-Hamiltonian

$$\tilde{H} = \tau + H = \tau + \frac{y^2}{2} - \frac{1}{u} - p(t)u.$$

The initial time variable $t \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is now an angle variable in this system and the variable τ, conjugate to t, is just the negative of the energy of H.

Remark 2.1. Indeed, since the Hamiltonian is merely C^1, the Peano existence theorem guarantees the existence of solutions of the Cauchy problem of the corresponding Hamiltonian equation but there is no guarantee for uniqueness. Nevertheless, for this particular system, after restricting to $\{\tilde{H} = 0\}$, we observe that omitting the equation of $\dot{\tau}$, the Hamiltonian equations are equivalent up to a translation of the time origin to the Hamiltonian equations associated to $H(u, y, t)$ for which the Cauchy–Lipschitz theorem guarantees the uniqueness of solutions. Thus, as the negative of the energy, τ is uniquely determined.

Remark 2.2. We remark that the dynamics in different energy surfaces of $\tilde{H}$ are all equivalent, since we may deduce one from the other simply by a shift of τ.

On $\{\tilde{H} = 0\}$, we change time (which now has nothing to do with the real time t) by multiplying $\tilde{H}$ by u and pulling back the resulting function by the usual symplectic 2-to-1 Levi-Civita transformation restricted to $T^*(\mathbb{R} \setminus \{z = 0\})$, i.e.,

$$\begin{aligned} \text{L.C.} : T^*(\mathbb{R} \setminus \{z = 0\}) \times T^*\mathbb{R} &\to T^*\mathbb{R}_+ \times T^*\mathbb{T}, \\ (z, w, t, \tau) &\mapsto \left(u = z^2, y = \frac{w}{2z}, t, \tau\right). \end{aligned}$$

The system $\tilde{H}$ is thus transformed into the system

$$K(z, w, t, \tau) = \text{L.C.}^*(u\tilde{H}) = z^2\tau + \frac{w^2}{8} - p(t)z^4 - 1,$$

which defines a Hamiltonian system on the symplectic manifold

$$(T^*(\mathbb{R} \setminus \{z = 0\}) \times T^*\mathbb{T}, dw \wedge dz + d\tau \wedge dt).$$

By its expression, this system can be extended analytically to

$$(T^*\mathbb{R} \setminus \{z = 0, w = 0\} \times T^*\mathbb{T}, dw \wedge dz + d\tau \wedge dt).$$

The extended system is regular on the two-dimensional subset $\mathcal{C}ol := \{z = 0\}$ of $\{K = 0\}$, which is called the *collision set*. As a matter of easy computation, we see that $|w| = 2\sqrt{2}$ on $\mathcal{C}ol$.

3 The Poincaré Section Related to the Collision Set

We now analyze, in our setting, the surface of section associated to collisions used in [5] and we give an alternative approach to [5, Section 6] concerning the exact symplecticity of this mapping.

For $\tau > 0$, when $p(t) = 0$, the dynamics of $K_0 = z^2\tau + w^2/8 - 1$ is easily understood. This is a coupled system of a harmonic oscillator in the (z, w)-space together with a rotator in (τ, t)-space. The set $\mathcal{C}ol^+ := \mathcal{C}ol \cap \{\tau > 0\}$ is seen to be a global section for the flow restricted to $\{K_0 = 0, \tau > 0\}$.

The Poincaré map $\mathcal{S} : \mathcal{C}ol^+ \subset \mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ sends a pair of collision time-energy (t_0, τ_0) to the successive pair of collision time-energy (t_1, τ_1). In [5, Proposition 5.1], the author has proved that there exists a 2π-periodic, lower semi-continuous function $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that the mapping is well-defined (i.e., $t_1 < +\infty$) on

$$D := \{(t_0, \tau_0) \in \tilde{D} : \tau_0 < \phi(t_0)\}$$

with $\min_{\mathbb{R}} \phi \geq -2\|p\|_\infty^{1/2}$.

We now follow [7] to show that this map is exact symplectic. For any small disk α in D and its image $\alpha' = \mathcal{S}(\alpha)$, we denote by Σ the cylinder formed by the integral curves of K connecting $\partial\alpha$ and $\partial\alpha'$. By the Stokes formula and the closeness of $\omega = d\lambda$ with $\lambda = w dz + \tau dt$, we have

$$\int_\alpha \omega = \int_{\alpha'} \omega + \int_\Sigma \omega.$$

Since Σ is formed by the integral curves of K and is two-dimensional, we deduce from the fact that the vector field X_K lies in the kernel of the restriction of ω to $\{K = 0\}$ that the further restriction of ω to Σ vanishes. Therefore,

$$\int_\alpha d\tau \wedge dt = \int_{\alpha'} d\tau \wedge dt.$$

On the other hand, for any loop γ in D and $\gamma' = \mathcal{S}(\gamma)$, we deduce by the Stokes formula that

$$\int_\gamma \lambda - \int_{\gamma'} \lambda = \int_\Sigma \omega = 0$$

and, therefore,

$$\int_\gamma \tau dt = \int_{\gamma'} \tau dt.$$

We thus have the following proposition.

Proposition 3.1. *The mapping $\mathcal{S}$ is exact symplectic with respect to the 2-form $d\tau \wedge dt$ on D.*

4 Some Periodic and Quasi-Periodic Solutions

Given the function $p(t)$, when restricted to a region of the phase space of K where z is small enough and $\tau > 0$ is bounded away from zero, the system $K(z, w, t, \tau)$ is a perturbation of the integrable system

$$K_0 = z^2\tau + \frac{w^2}{8} - 1.$$

The “integrable approximating system” K_0 reads

$$\frac{\sqrt{2}}{2}\tau^{1/2}I - 1$$

in action-angle form by calculating the action-angle coordinates (I, θ, τ, t') defined by the relations

$$\begin{cases} z = 2^{-1/4}I^{1/2}\tau^{-1/4}\cos\theta, \\ w = -2\cdot 2^{1/4}I^{1/2}\tau^{1/4}\sin\theta, \\ \tau = \tau, \\ t = t' + 4^{-1}I\tau^{-1}\sin 2\theta. \end{cases}$$

The calculation goes in the following way. We first fix τ and reduce the system K_0 by the $\mathbb{S}^1$-symmetry of shifting t, and then we calculate the action variable $I = A(h)/2\pi$ by calculating the enclosed area $A(h)$ of the ellipse $\{K_0 = h\}$ in the (z, w)-plane. Inverting the mapping $h \mapsto I(h)$ gives

$$K_0 = h = \frac{\sqrt{2}}{2}\tau^{1/2}I - 1.$$

We thus set

$$z = 2^{-1/4}I^{1/2}\tau^{-1/4}\cos\theta, \quad w = -2\cdot 2^{1/4}I^{1/2}\tau^{1/4}\sin\theta.$$

In the unreduced phase space with variables (I, θ, τ, t), we have

$$dw\wedge dz + d\tau\wedge dt = dI\wedge d\theta + d\tau\wedge d(t - 4^{-1}I\tau^{-1}\sin 2\theta).$$

Therefore, the set of variables $(I, \theta, \tau, t' = t - 4^{-1}I\tau^{-1}\sin 2\theta)$ forms a set of action-angle coordinates of K_0.

The subset $\{\tau > 0, I > 0\}$ of the phase space of K_0 is seen to be foliated by the invariant 2-tori of K_0 obtained by fixing I and τ, and the associated Poincaré map $\check{\mathbb{S}} = \mathbb{S}|_D$ is an exact twist map. The perturbation takes the form

$$P(I, \theta, \tau, t') = p\big(t(I, \theta, \tau, t')\big)I^2\tau^{-1}\cos^4\theta.$$

We now have a choice to study the dynamics of K as a perturbation of K_0 (which is the case for given $p(t)$ when $|z|$ is small enough). We may either work directly with the Hamiltonian or with the Poincaré map $\check{\mathbb{S}}$. The objects between the two approaches are naturally related. Fixed and periodic points of $\check{\mathbb{S}}$ give rise to periodic solutions of K and invariant curves of $\check{\mathbb{S}}$ give rise to invariant tori of K. They give rise to generalized periodic and quasi-periodic solutions of (1.1), respectively. Recall that by “generalized solutions” we simply refer to those collision solutions along which the collisions are regularized by elastic bouncing, which is exactly what the Levi-Civita regularization implies for the initial system.

We now prove Theorem 1.1.

Proof of Theorem 1.1. We opt to work with the Hamiltonian function. The reader is invited to compare with [5] for results obtained from analyzing the mapping $\check{\mathbb{S}}$.

We introduce a small parameter ε and we shall apply the KAM theorem to invariant tori of K_0 in a region where $\tau \sim \varepsilon^{-2}$, $I \sim \varepsilon$, and where the unperturbed energy of K_0 lies in $(-\delta, \delta)$ for a certain small $\delta > 0$ independent of ε.

We observe that the unperturbed system K_0 is non-degenerate in the sense that its Hessian with respect to τ and I is non-degenerate. Indeed, since K_0 depends linearly on I, we have $\partial^2 K_0/\partial I^2 = 0$ and the determinant of this Hessian matrix equals $-(\partial^2 K_0/\partial I\partial\tau)^2$, which, up to a constant, is equal to $\tau^{-1} \sim \varepsilon^2$.

Since the non-degeneracy of the unperturbed system depends non-trivially on the small parameter ε, in order to show that a standard KAM theorem holds (a simple version that applies to our case is [1, Theorem 6.16], in which it is also remarked that it is enough to have $r > 4$ to apply [6, Theorem 2.1]), we have to make yet another rescaling. Set

$$(I, \theta, \tau, t') = (\varepsilon^{-2}I', \theta, \varepsilon^{-2}\tau', t').$$

We have

$$dI\wedge d\theta + d\tau\wedge dt' = \varepsilon^{-2}(dI'\wedge d\theta + d\tau'\wedge dt'), \quad K_0(I, \tau) = \varepsilon^{-3}K_0(I', \tau').$$

Moreover, by its explicit expression, we also have

$$P(I, \theta, \tau, t') = \varepsilon^{-2} P(I', \theta, \tau', t').$$

The rescaled complete system is thus equivalent to the system with Hamiltonian $K_0(I', \tau') + \varepsilon P(I', \theta, \tau', t')$ and the (standard) symplectic form $dI' \wedge d\theta + d\tau' \wedge dt$ after a further rescaling of time. In the region $\tau' \sim 1$, $I' \sim \varepsilon^3$, the unperturbed function $K_0(I', \tau')$ is C^∞-smooth. The determinant of the Hessian of $K_0(I', \tau')$ reads $-(\partial^2 K_0(I', \tau')/\partial I' \partial \tau')^2 \sim 1$ and is now independent of the small parameter ε. For $r > 4$, the C^r-norm of the perturbation $\varepsilon P(I', \theta, \tau', t')$ is seen to be of order $O(\varepsilon)$. The cited KAM theorem can thus be applied directly to this rescaled system, provided that $\varepsilon > 0$ is chosen to be small enough.

We thus find a set of positive measure of invariant tori of K, in particular, the function K takes values in $(-\delta/2, \delta/2)$ on each of them, provided ε is small enough, and we deduce from [6, Theorem 2.1] that when $r > 9$, these invariant tori are accumulated by periodic orbits of K. In view of Remark 2.2 and by Fubini's theorem, the persisted invariant 2-tori form a set of positive measure in $\{K = 0\}$ accumulated by periodic orbits, provided that ε is small enough. By assumption, $\tau \to \infty$ when $\varepsilon \to 0$. □

Remark 4.1. We may also impose a smallness condition (in addition to the smoothness condition) on $p(t)$ to find invariant tori of K occupying a larger set in the phase space by an application of the KAM theorem. The proof goes along the same lines except for the fact that it is no longer necessary to rescale the action variables.

The existence of these two-dimensional KAM tori in the three-dimensional $\{K = 0\}$ entails that these periodic orbits are "stable" in the sense that there are no large changes of the action variables τ and I, which translates into the energy and the amplitude of (1.1), respectively.

In [5], the existence of families of generalized periodic solutions is shown by an application of the Poincaré–Birkhoff theorem. In our case, these periodic solutions, which correspond to periodic points of the Poincaré map lying between invariant curves (which are themselves intersections of invariant 2-tori with the constructed Poincaré section), are stable à la Lagrange in the same sense that there are no large changes of the energy and the amplitude. Moreover, these invariant curves bound the positive but finite $|d\tau \wedge dt|$-measure in D preserved by $\mathcal{S}$, which allows us to apply the Poincaré recurrence theorem (see [1, Section 2.6]) to confirm that $|d\tau \wedge dt|$-almost all points in the bounded regions (what is called Birkhoff's region of instability) are recurrent. Therefore, the dynamics confined to these bounded regions is Poisson stable.

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References

[1] V. I. Arnold, V. V. Kozlov and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, 3rd ed., Encyclopaedia Math. Sci. 3, Springer, Berlin, 2006.
[2] E. Goursat, Les transformations isogonales en mécanique, *C. R. Math. Acad. Sci. Paris* **108** (1887), 446–450.
[3] A. Lazer and S. Solimini, On periodic solutions of nonlinear differential equations with singularities, *Proc. Amer. Math. Soc.* **99** (1987), no. 1, 109–114.
[4] T. Levi-Civita, Sur la régularisation du probleme des trois corps, *Acta Math.* **42** (1920), no. 1, 99–144.
[5] R. Ortega, Linear motions in a periodically forced Kepler problem, *Port. Math.* **68** (2011), no. 2, 149–176.
[6] J. Pöschel, *Über invariante Tori in differenzierbaren Hamiltonschen Systemen*, Bonn. Math. Schr. 120, Universität Bonn, Bonn, 1980.
[7] D. Treschev and O. Zubelevich, *Introduction to the Perturbation Theory of Hamiltonian Systems*, Springer Monogr. Math., Springer, Berlin, 2009.